Journal of Engineering Mathematics 20 (1985) 171–179

On dual solutions occurring in mixed convection in a porous medium

J.H. MERKIN

Department of Applied Mathematical Studies, University of Leeds, Leeds LS2 9JT, England

(Received October 4, 1985)

Summary

The dual solutions to an equation, which arose previously in mixed convection in a porous medium, occurring for the parameter α in the range $0 < \alpha < \alpha_0$ are considered. It is shown that the lower branch of solutions terminates at $\alpha = 0$ with an essential singularity. It is also shown that both branches of solutions bifurcate out of the single solution at $\alpha = \alpha_0$ with an amplitude proportional to $(\alpha_0 - \alpha)^{1/2}$. Then, by considering a simple time-dependent problem, it is shown that the upper branch of solutions is stable and the lower branch unstable, with the change in temporal stability at $\alpha = \alpha_0$ being equivalent to the bifurcation at that point.

1. Introduction

In a previous paper [1] the author obtained the equation

$$F''' + FF'' = 0, \quad F(0) = 0, \quad F'(0) = -\alpha, \quad F' \to 1 \quad \text{as } y \to \infty, \tag{1}$$

(primes denote differentiation with respect to the independent variable y) in the context of mixed-convection boundary-layer flow in a saturated porous medium. It was shown in [1] that this equation had just one solution for $\alpha \leqslant 0$ and no solutions for $\alpha > \alpha_0$ ($\alpha_0 \simeq 0.354$), while for $0 < \alpha < \alpha_0$ there were two solutions, an upper solution F_u and a lower solution F_l with $0 < F_l''(0) < F_u''(0)$. It is the purpose of this paper to complete the discussion of equation (1) by considering the behaviour of the lower solution as $\alpha \to 0$ from above (putting $\alpha = 0$ gives the Blasius solution for the upper solution) and the nature of the two solutions near $\alpha = \alpha_0$.

The first case is similar in some respects to the behaviour of the reversed-flow solutions of the Falkner-Skan equation as the Falkner-Skan parameter $\beta \to 0$ from below, as treated by Brown and Stewartson [2]. In both cases the solution divides up into two regions, a thick inviscid inner region and a much thinner outer shear layer, but the details for the two problems are somewhat different. We show that F_l has an essential singularity at $\alpha = 0$, with, for example, $F_l''(0)$ being of $O(\exp(-a_0^2/2\alpha))$ for small α (a_0 is a constant determined in the solution). In the second case we find that the perturbation to the solution at $\alpha = \alpha_0$ is of $O((\alpha_0 - \alpha)^{1/2})$. The upper and lower solutions bifurcate out of the single solution at $\alpha = \alpha_0$ with the upper solution arising from the positive sign and the lower one from the negative sign obtained on taking a square root.

A further question that arises is which of the two solutions F_u or F_l will be obtained in practice; since $F_l''(0)$ and $F_u''(0)$ are both positive they are each physically acceptable solutions for the original heat-transfer problem. To this end we consider the simple time-dependent problem

$$\frac{\partial^2 f}{\partial y\, \partial t} = \frac{\partial^3 f}{\partial y^3} + f\frac{\partial^2 f}{\partial y^2} \tag{2}$$

with boundary conditions (for $t > 0$)

$$f = 0, \quad \frac{\partial f}{\partial y} = -\alpha \quad \text{on } y = 0; \quad \frac{\partial f}{\partial y} \to 1 \quad \text{as } y \to \infty, \tag{3}$$

and initial condition

$$f = 0 \quad \text{at } t = 0 \text{ (for } y > 0\text{)}. \tag{4}$$

Numerical integrations of equation (2) for a range of α in $0 < \alpha < \alpha_0$ show that, in each case, the solution $f(y, t)$ approaches $F_u(y)$ as $t \to \infty$. The reason for this becomes clear when we consider the behaviour of $f(y, t)$ for large t. We find that $f(y, t) - F(y)$ is of $O(e^{-\lambda t})$ with λ given by a linear eigenvalue problem (involving F_u or F_l). A numerical computation of the smallest eigenvalue λ_1 then shows that λ_1 is always positive for F_u and always negative for F_l, so that, for this simple problem at least, F_u is the stable and F_l the unstable solution, with it being possible to reach only F_u from some initial configuration.

2. The lower solution as $\alpha \to 0^+$

Numerical solutions of equation (1) indicate that, as $\alpha \to 0^+$, $F_l''(0) \to 0$ with there being a thick inner region in which $F_l \simeq -\alpha y$ and a much thinner outer region in which F_l' changes from a small negative value to satisfy the outer boundary condition that $F' \to 1$. This suggests expanding F in the inner region as

$$F = -\alpha y + A\phi_1(y) + A^2\phi_2(y) + \ldots \tag{5}$$

where $A = A(\alpha)$ is small and whose form will be determined from the matching with the outer solution. On substituting (5) into equation (1) and equating like powers, we obtain the equation for ϕ_1 as

$$\phi_1''' - \alpha y \phi_1'' = 0, \tag{6}$$

which has the solution satisfying $\phi_1(0) = \phi_1'(0) = 0$ (the outer boundary condition is relaxed at this stage)

$$\phi = A_1\left[y\int_0^y e^{\alpha s^2/2}\, ds + \alpha^{-1}\left(1 - e^{\alpha y^2/2}\right)\right] \tag{7}$$

where A_1 is a constant. For $y \gg 1$, (7) gives

$$\phi_1 \sim A_1\, e^{\alpha y^2/2}\left(\frac{1}{\alpha^2 y^2} + \frac{3}{\alpha^3 y^4} + \ldots\right). \tag{8}$$

The equation for ϕ_2 is linear and can be solved by quadratures; the details are lengthy and need not be given. All we require here is the behaviour of ϕ_2 for large y and we find that

$$\phi_2 \sim -\frac{A_1^2}{8\alpha^5}\,e^{\alpha y^2}\left(\frac{2}{y^5}+\frac{21}{\alpha y^7}+\ldots\right), \tag{9}$$

so that

$$F \sim -\alpha y + A(\alpha)\frac{A_1\,e^{\alpha y^2/2}}{\alpha^2 y^2} - A^2(\alpha)A_1^2\frac{e^{\alpha y^2}}{4\alpha^5 y^5}+\ldots \tag{10}$$

for $y \gg 1$.

Expansion (5) is an inner solution and we require a further outer region which matches to (10) and at the outer edge of which the outer boundary condition is satisfied. A consideration of the terms in equation (1) shows that the only possibility for this outer solution is for F to be of O(1) and its thickness also to be of O(1). Then, since expansion (10) giving F at the outer edge of the inner region has to match to this outer solution, it will break down when F is of O(1), i.e. when y is of $O(\alpha^{-1})$. This suggests defining the independent variable ζ for the outer region by

$$y = \frac{a_0}{\alpha} + \zeta \tag{11}$$

where a_0 is a constant which will be determined by the matching. This leaves equation (1) unaltered (though primes now denote differentiation with respect to ζ) together with the boundary condition

$$F' \to 1 \quad \text{as } \zeta \to \infty. \tag{12}$$

Now, at $y = a_0/\alpha$, (10) gives

$$F \sim -a_0 + A(\alpha)\frac{A_1\,e^{a_0^2/2\alpha}}{a_0^2} - A(\alpha)^2\frac{A_1^2\,e^{a_0^2/\alpha}}{4a_0^5}+\ldots, \tag{13}$$

and so for F to remain of O(1) we must have

$$A(\alpha) = e^{-a_0^2/2\alpha}. \tag{14}$$

This fixes $A(\alpha)$ and with this form for A, using (11), we obtain the inner condition for the outer region that

$$F \sim \left(-a_0 + \frac{A_1}{a_0^2}e^{a_0\zeta} - \frac{A_1^2}{4a_0^5}e^{2a_0\zeta}+\ldots\right) + \alpha\left(-\zeta + \frac{A_1}{a_0^2}e^{a_0\zeta}\left(\frac{\zeta^2}{2} - \frac{2\zeta}{a_0} + \frac{3}{a_0^2}\right)+\ldots\right) \tag{15}$$

as $\zeta \to -\infty$.

(15) suggests an expansion for F in the outer region as

$$F = F_0 + \alpha F_1 + \dots \tag{16}$$

where F_0 satisfies equation (1) together with the boundary condition (12) and

$$F_0 \sim a_0 + \frac{A_1}{a_0^2} e^{a_0 \zeta} - \frac{A_1^2}{4 a_0^5} e^{2 a_0 \zeta} + \dots \tag{17}$$

as $\zeta \to -\infty$. This problem has already been considered by Chapman [3] and its numerical solution determines a_0 as $a_0 = 0.8758$. Further we can check that the asymptotic expansion (17) for F_0 agrees with the asymptotic expansion derived from the equation for F_0 as $\zeta \to -\infty$, and by comparing the numerical solution with (17) we can estimate a value for A_1 as $A_1 \simeq 1.24$. One point to note about this solution is that it is not unique, being unaltered by an O(1) shift in origin in ζ, say by a_1. This reflects the fact that the outer region should be centred on $y = (a_0/\alpha) + a_1 + \dots$, not on $y = a_0/\alpha$ exactly.

The equation for the term of $O(\alpha)$ is

$$F_1''' + F_0 F_1'' + F_1 F_0'' = 0 \tag{18}$$

with boundary conditions

$$\begin{aligned} &F_1' \to 0 \quad \text{as } \zeta \to \infty, \\ &F_1 \sim -\zeta + \frac{A_1}{a_0^2} e^{a_0 \zeta} \left(\frac{\zeta^2}{2} - \frac{2\zeta}{a_0} + \frac{3}{a_0^2} \right) + \dots \quad \text{as } \zeta \to -\infty. \end{aligned} \tag{19}$$

The solution of equation (18) determines a_1 as a change from ζ to $\bar{\zeta} = \zeta + a_1$ leaves the problem unaltered if we solve equation (18) subject to the condition that $F_1' \to -1$ as $\zeta \to -\infty$. This gives $a_1 = 2.704$.

Finally, we have that as $\alpha \to 0^+$

$$F_l''(0) = A_1 \, e^{-a_0^2/2\alpha} + \dots . \tag{20}$$

To compare $F_l''(0)$ as given by (20) with values obtained from the numerical solutions of equation (1) we calculate the quantity $q = F_l''(0) \exp(a_0^2/2\alpha)$ and values of q for a range

Table 1. Values of $F_l''(0)$ and $q = F_l''(0) \exp(a_0^2/2\alpha)$, with $a_0 = 0.8758$, for various α

α	$F_l''(0)$	q
0.16	$1.0749.\ 10^{-2}$	0.1181
0.14	$6.8338.\ 10^{-3}$	0.1058
0.12	$3.9346.\ 10^{-3}$	0.0961
0.10	$1.9356.\ 10^{-3}$	0.0896
0.09	$1.2382.\ 10^{-3}$	0.0878
0.08	$7.2152.\ 10^{-4}$	0.0871
0.07	$3.6698.\ 10^{-4}$	0.0879

of α are given in Table 1. q should approach a constant value A_1 as $\alpha \to 0$ and it appears to be doing so, with $A_1 \simeq 0.087$, given the difficulty in obtaining accurate numerical solutions for small values of α when $F_l''(0)$ is getting extremely small. This value of A_1 appears to be at odds with the value of A_1 estimated from the solution in the outer region. This discrepancy is explained by noting that the form for F_0 as $\zeta \to -\infty$ in the outer region should really be that $F_0 \sim -a_0 + (A_1/a_0^2)\,\mathrm{e}^{a_0\bar{\zeta}} + \ldots$, so that to make the comparison we should replace A_1 by $A_1\,\mathrm{e}^{-a_1} \simeq 0.083$ and then the agreement becomes clearer, given the difficulty in estimating A_1 accurately from the numerical solution.

3. The solution as $\alpha \to \alpha_0$

Consider some small perturbation ϕ to the solution F_0 of equation (1) for a particular value of α. Then ϕ will satisfy the equation (after linearising)

$$\phi''' + F_0\phi'' + F_0''\phi = 0 \tag{21}$$

with boundary conditions

$$\phi(0) = \phi_0'(0) = 0,\ \phi' \to 0 \quad \text{as } y \to \infty. \tag{22}$$

(21) and (22) form a homogeneous problem and will, in general, have only the trivial solution $\phi \equiv 0$. (Since any constant multiple of ϕ is also a solution, we can take $\phi''(0) = 1$ without loss of generality and then solve equation (21) as an initial-value problem. There is then no reason why, that for a general value of α, ϕ' should tend to zero).

However, we can regard the obtaining of a non-trivial solution to equation (21) (together with equation (1)) satisfying (22) as an eigenvalue problem for determining α_0 (the value of α which limits the range of the dual solutions). A numerical integration, fixing $\phi''(0) = 1$ and regarding α as a parameter, gives $\alpha_0 = 0.35411$ with a corresponding $F_0''(0) = 0.21785$, in agreement with the previously calculated solutions given in [1]. This is the crucial step in determining the behaviour of the soluton near $\alpha = \alpha_0$, for we now take $\alpha = \alpha_0 - \epsilon$ where $\epsilon > 0$ and is assumed small and expand F as

$$F = F_0 + \epsilon^{1/2}F_1 + \epsilon F_2 + \ldots \tag{23}$$

where F_0 satisfies equation (1) with $\alpha = \alpha_0$. F_1 satisfies equation (21) and boundary conditions (22) which has now a non-trivial solution, namely $F_1 = K\phi$ for some constant K. F_2 then satisfies the equation

$$F_2''' + F_0F_2'' + F_0''F_2 = -K^2\phi\phi'' \tag{24}$$

with boundary conditions

$$F_2(0) = 0,\ F_2'(0) = 1,\ F_2' \to 0 \quad \text{as } y \to \infty. \tag{25}$$

Equation (24) is solved by constructing two particular integrals, F_a and F_b, where F_a is obtained by integrating equation (24) with $K = 1$ and $F_a'(0) = 0$, $F_a''(0) = 0$, and F_b by integrating equation (24) with $K = 0$, $F_b'(0) = 1$, $F_b''(0) = 0$. The general solution is then

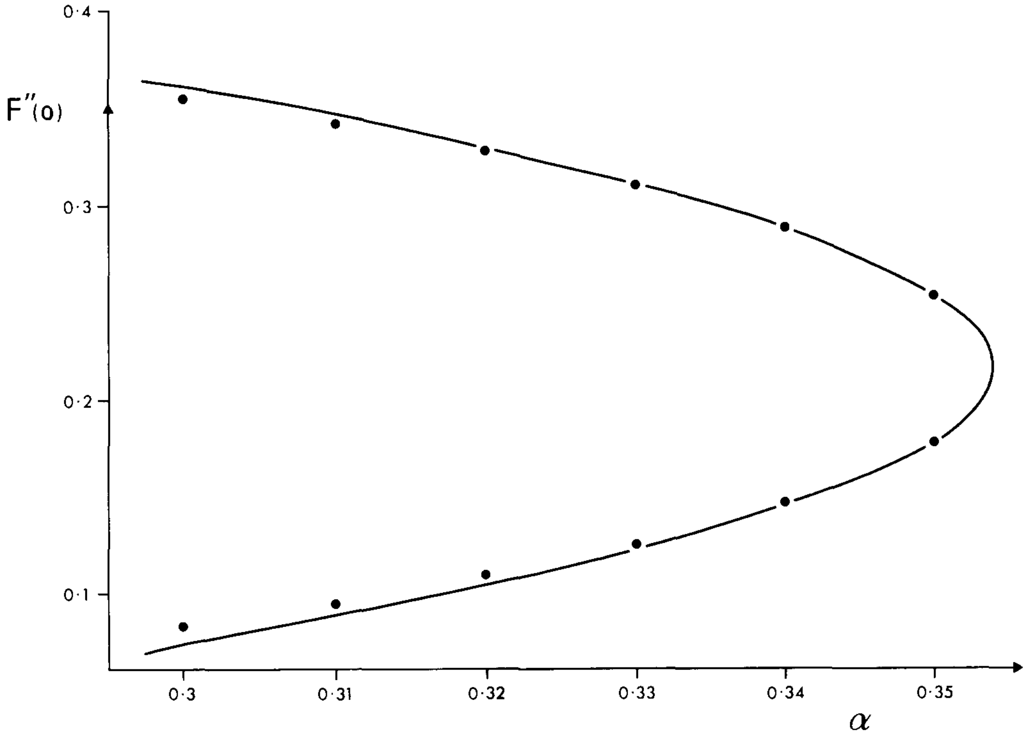

Figure 1. A graph of $F''(0)$ for α close to α_0 as calculated from (27), with values determined from a numerical integration of equation (1) shown by ●.

given by $F_2 = F_b + K^2\ F_a$. (This solution is not unique as we can always add on an arbitrary multiple of ϕ). We can easily show that, as $y \to \infty$, $F_i' \to C_i$ $(i = a, b)$ for any general integration of (24), with C_i a constant. Then, as $y \to \infty$, $F_2' \to C_b + K^2 C_a$ and to satisfy the outer boundary condition we must take $K^2 = -C_b/C_a$. The numerical integrations give $C_a = 7.3672$, $C_b = -2.8240$ and hence $K = \pm 0.6191$. From which it follows that

$$F = F_0 \pm 0.6191(\alpha_0 - \alpha)^{1/2}\phi + O((\alpha_0 - \alpha)). \tag{26}$$

The positive sign in (26) is the start of the branch of upper solutions F_u and the negative sign the start of the lower branch F_l. In particular we have

$$F''(0) = 0.21785 \pm 0.6191(\alpha_0 - \alpha)^{1/2} + \dots. \tag{27}$$

Figure 1 gives a graph of $F''(0)$ as calculated from (27) and also values obtained by solving equation (1) numerically. These are in excellent agreement close to $\alpha = \alpha_0$, giving a satisfactory confirmation of the above theory.

4. Time-dependent problem

We now consider the solution of equation (2) subject to boundary conditions (3) and initial conditions (4). This problem is similar to one treated previously by Ingham,

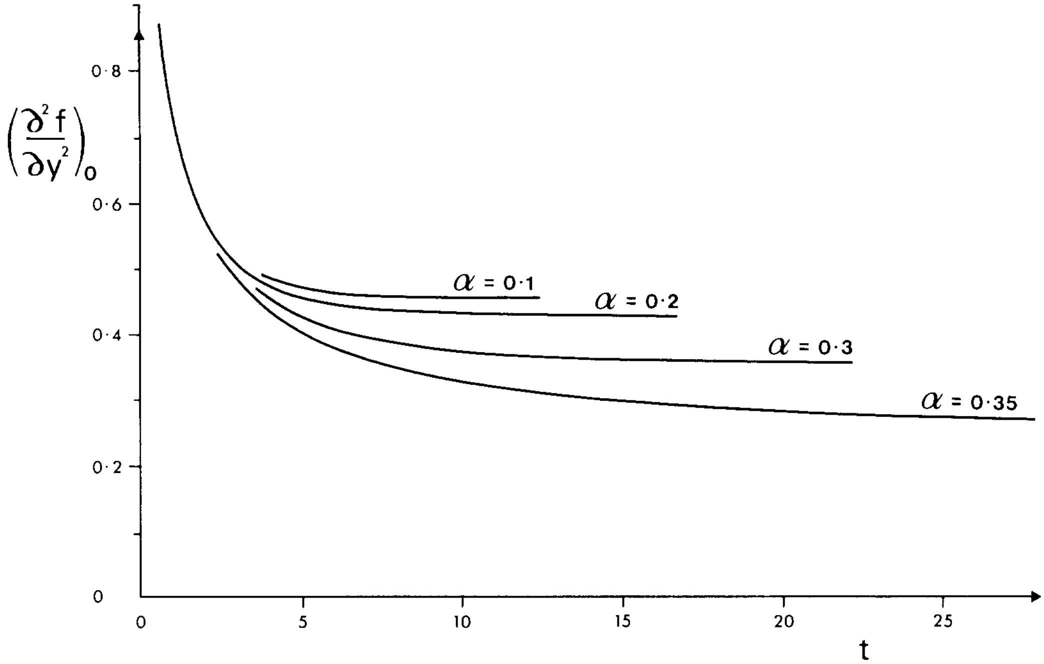

Figure 2. Graphs of $(\partial^2 f/\partial y^2)_{y=0}$ against t calculated from the numerical integration of equation (2) for $\alpha = 0.1$, 0.2, 0.3 and 0.35.

Merkin and Pop [4] (in fact the equation is the same though the boundary conditions are different) and was solved numerically in the same way as described in [4]. It is not necessary to repeat all the details of the numerical scheme here, noting only that a transformed version of the equation, with $f = t^{1/2}h(\eta, t)$, $\eta = y/t^{1/2}$, was used to start the integration, the change to equation (2) being made at $t = 1$. Values of $(\partial^2 f/\partial y^2)_{y=0}$ thus obtained for $\alpha = 0.1$, 0.2, 0.3 and 0.35 are shown in Figure 2. In each case we can see that $(\partial^2 f/\partial y^2)_{y=0}$ approaches the value of $F_u''(0)$ corresponding to that particular value of α as $t \to \infty$. Other values α in $0 < \alpha < \alpha_0$, as well as different initial conditions, were also tried and the same conclusion was reached in each case.

To see why this is the case, consider the behaviour of $f(y, t)$ as $t \to \infty$ for α in $0 < \alpha < \alpha_0$. To do this, put $f(y, t) = F_0(y) + G(y, t)$ where F_0 is the corresponding solution of equation (1) and where, for $t \gg 1$, G is small and satisfies the (linearised) equation

$$\frac{\partial^3 G}{\partial y^3} + F_0 \frac{\partial^2 G}{\partial y^2} + G F_0'' = \frac{\partial^2 G}{\partial y\, \partial t}. \tag{28}$$

From which it follows, by a separation-of-variables argument, that $G = e^{-\lambda t} g(y)$, for some constant λ, with g satisfying the equation

$$g''' + F_0 g'' + F_0'' g + \lambda g' = 0, \tag{29}$$

$$g(0) = 0, \quad g'(0) = 0, \quad g' \to 0 \quad \text{as } y \to \infty, \tag{30}$$

(the form for G above may be multiplied by some term algebraic in t but this will not affect equation (29), not alter the conclusions on the stability of F_0 deduced from the sign of λ).

For $y \gg 1$, $F_0 \sim y + \delta_0 +$ exponentially small terms, so that equation (29) becomes, approximately,

$$g''' + \bar{y}g'' + \lambda g' = 0 \tag{31}$$

where $\bar{y} = y + \delta_0$. Equation (31) can be solved in terms of confluent hypergeometric functions, [5], from which we have that

$$g' \sim D\bar{y}^{-\lambda} + E\bar{y}^{\lambda-1}\, \mathrm{e}^{-\bar{y}^2/2}. \tag{32}$$

We require solutions with exponential decay, for we expect the algebraic decay terms to lead to difficulties in the higher-order terms [6] i.e. we have to find that non-trivial solution of equation (29) subject to boundary condition (30) which has $D = 0$. This has to be done numerically and we have computed the least eigenvalue λ_1 for a range of α in $0 < \alpha < \alpha_0$ for both upper and lower solutions. These values are shown in Table 2. We can see that $\lambda_1 > 0$ for the upper solutions and $\lambda_1 < 0$ for the lower solutions so that the upper branch of solutions is stable and the lower branch unstable and hence these are not physically realisable from some initial configuration. To confirm this, equation (2) was solved using $F_l(y)$ as initial data. Though this is an exact solution of (2), the small errors introduced by the numerical scheme were sufficient to move this solution away from F_l

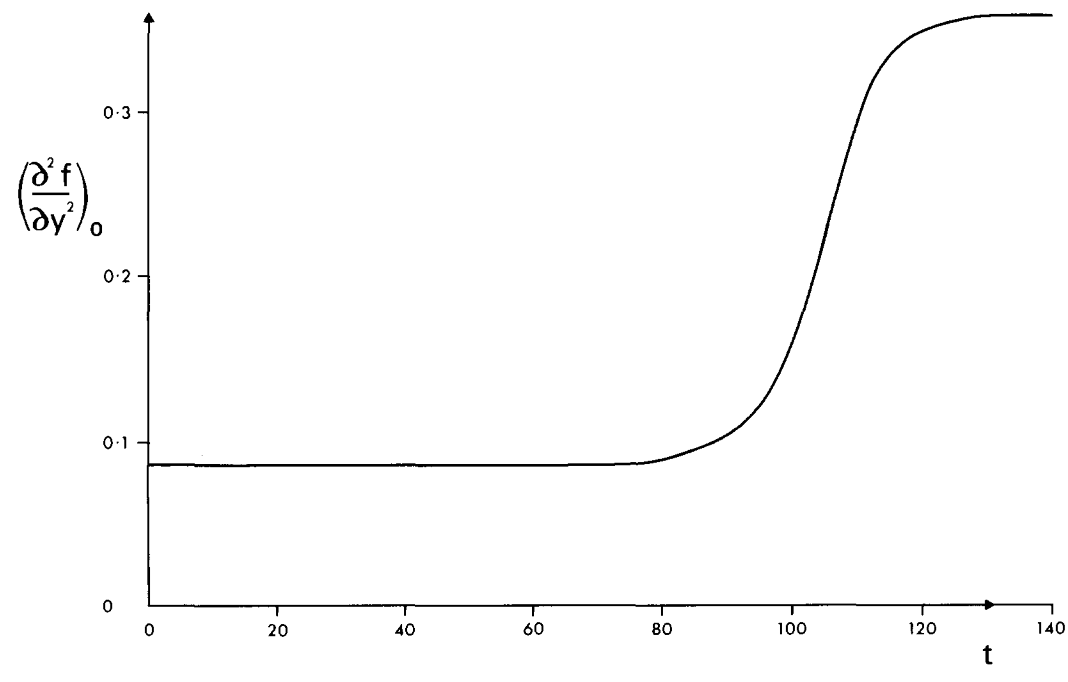

Figure 3. A graph of $(\partial^2 f/\partial y^2)_{y=0}$ against t for $\alpha = 0.3$ found by solving equation (2) numerically starting with $F_l(y)$ as initial data.

Table 2. Values of the smallest eigenvalue λ_1 for various α

α	Upper solution	Lower solution
0.0	0.8096	–
0.1	0.6508	−0.0917
0.2	0.4715	−0.1413
0.3	0.2470	−0.1332
0.32	0.1879	−0.1164
0.34	0.1134	−0.0839
0.35	0.0577	−0.0492

and onto F_u. The results of this integration (for $\alpha = 0.3$) are shown in Figure 3. Also, as α increases, the value of λ_1 decreases so that the rate of approach to the steady state becomes slower, as is confirmed by the plots of $(\partial^2 f/\partial y^2)_{y=0}$ shown in Figure 2.

Equation (21) for determining α_0, the value of α from which the two branches of solutions bifurcate, is just equation (29) with $\lambda = 0$. This latter equation was used to find λ and hence to determine the temporal stability of each of the two branches. So that the change in stability at $\alpha = \alpha_0$ and the bifurcation at this point can be regarded as alternative ways of looking at the same effect, and since this latter is dependent only on the steady-state equation (1) we expect the conclusion that the upper branch of solutions is stable and the lower branch unstable to hold for a general time-dependent problem for which these solutions are possible steady states.

References

[1] J.H. Merkin, Mixed convection boundary layer flow on a vertical surface in a saturated porous medium, *J. Engng. Math.* 14 (1980) 301–313.

[2] S.N. Brown and K. Stewartson, On the reversed flow solutions of the Falkner-Skan equation, *Mathematika* 13 (1966) 1–6.

[3] D.R. Chapman, Laminar mixing of a compressible fluid, NACA Rep. 958 (1950).

[4] D.B. Ingham, J.H. Merkin and I. Pop, The collision of free-convection boundary layers on a horizontal cylinder embedded in a porous medium, *Q. Jl. Mech. Appl. Math.* 36 (1983) 313–335.

[5] L.J. Slater, *Confluent hypergeometric functions*, Cambridge University Press (1960).

[6] J.H. Merkin, On solutions of the boundary-layer equations with algebraic decay, *J. Fluid Mech.* 88 (1978) 309–321.